Example - 6
P-86

The Elastic fluid obeys Boyle's law

$$p = K\rho \quad \text{--- } ①$$

Equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \quad \text{--- } ②$$

Equation of motion

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{--- } ③$$

Eq^n. ① $\quad \frac{\partial p}{\partial x} = K \cdot \frac{\partial \rho}{\partial x}$

From ③ $\quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{K}{\rho}\frac{\partial \rho}{\partial x} \quad \text{--- } ④$

Differentiating ④ Partially with regard to t we have

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial t}\left(v\frac{\partial v}{\partial x} + \frac{K}{\rho}\frac{\partial \rho}{\partial x}\right) = 0$$

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial t} + \frac{K}{\rho}\frac{\partial \rho}{\partial t}\right) = 0$$

Eq^n ② $\quad \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v)$

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial x}\left[v\frac{\partial v}{\partial t} - \frac{K}{\rho}\frac{\partial}{\partial x}(\rho v)\right] = 0$$

So this becomes

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial x}\left[v\frac{\partial v}{\partial t} - K\frac{\partial v}{\partial x} - \frac{K}{\rho}\frac{\partial \rho}{\partial x}v\right] = 0$$

From ③

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{K}{\rho}\frac{\partial \rho}{\partial x}$$

C-3

Conservation of Momentum

The momentum of a body is defined as the Product of the mass of the body and its velocity

i.e. Velocity $= \frac{mq}{g_0}$

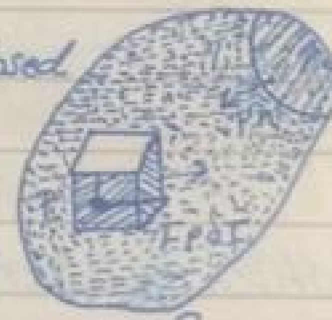
Has the dimensions of force-time In the flow of fluids the momentum M Per unit Volume is given By

$$M = \frac{\sigma q}{g_0} = \rho q$$

the velocity is a Vector quantity so Momentum is likewise a Vector quantity having magnitude and Both direction.

Equation of motion of an inviscid fluid:—

Consider any arbitrary closed surface S drawn in the region occupied by the incompressible fluid at an instant t.



ρ be the density of the fluid Particle at the Point P. with the closed surface And $d\tau$ we the volume of the fluid enclosing be the Point P.

Mass of the fluid element $= \rho\, d\tau$

= volume × density

Velocity Potential, Irrotational Flow:-

We know that the equation to stream-line are $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$ —— (1)

These curves cut the surface

$$u\,dx + v\,dy + w\,dz = 0 \quad \text{—— (2)}$$

orthogonally. Consider a scalar function $\phi(x, y, z, t)$ at that instant uniform throughout the motion such that

$$u\,dx + v\,dy + w\,dz = -d\phi$$

Neglecting sign is considerd as a matter of convention

$$u\,dx + v\,dy + w\,dz = -\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$\Rightarrow \boxed{u = -\frac{\partial\phi}{\partial x},\ v = -\frac{\partial\phi}{\partial y},\ w = -\frac{\partial\phi}{\partial z}}$$

$$\Rightarrow q = -\nabla\phi = -\text{grad}\,\phi$$

where ϕ (scalar function) is termed the velocity Potential

$$q = -\nabla\phi$$

$$\nabla\cdot q = -\nabla(\nabla\phi)$$

$$= -\nabla^2\phi = 0$$

$$\Rightarrow \nabla^2\phi = 0$$

$\Rightarrow$ ϕ is a harmonic function Satisfying laplace equation.

$\Rightarrow$ Velocity Potential is a harmonic function

Hence the velocity Potential is a scalar function of Space and time.

$$\nabla\times q = \nabla\times(-\nabla\phi) = -\nabla(\nabla\phi)$$

Boundary surface $\frac{\partial F}{\partial t} + \frac{\partial r}{\partial t} \nabla F = 0$

$$\Rightarrow \quad \frac{\partial F}{\partial t} + q \nabla F = 0$$

Equation of continuity to a moving Boundary surface

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

If the surface be at rest than $\frac{\partial F}{\partial t} = 0$

$$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

Normal velocity of the Boundary

$$u \cdot n = \frac{u \cdot \nabla F}{|\nabla F|}$$

$$\boxed{u \cdot n = \frac{\partial F / \partial t}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}}$$

—o—

Ex-36/63 Show that $\frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t = 1$

is a Possible form for the bounding surface of a liquid and find an expression for the normal velocity.

Sol.ⁿ The Boundary surface of a liquid is given as

$$F(x, y, z, t) = \frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t - 1 = 0$$

Which is remain constant through the motion and q be the velocity of the fluid Particle.
Then Moment = velocity x mass

$$M = \int q \rho \, d\tau$$

time rate of change of momentum

$$\frac{dM}{dt} = \int \frac{d}{dt}\{ q (\rho d\tau)\}$$

$$\frac{dM}{dt} = \frac{dq}{dt}(\rho d\tau) + \int q \cdot \frac{d}{dt}(\rho d\tau)\}$$

$$\frac{dM}{dt} = \int \frac{dq}{dt} \rho \, d\tau \quad \text{———} (1)$$

Here second integer vanish
Let F be the impressed force Per unit mass acting on fluid Particle at P.

Total force on the Volume $= \int F \rho \, d\tau$ —— (2)

Let P be the Pressure at a Point on the surface along the outward drawn unit normal $\hat{n}$ then the force on the fluid Particle due to the action of the surrounding fluid is

$$= -\int P\hat{n} \, ds$$

$$= -\int \nabla P \, d\tau \qquad \text{By Gauss' theorem}$$

the equation of momentum

+ Rate of momentum accumulation = Rate of momen^m - Rate of momentum out + Sum of force acting on syst.

$$\int \rho \frac{dq}{dt} d\tau = \int F \rho \, d\tau - \int \nabla P \, d\tau$$

or

$$\int \left[\rho \frac{dq}{dt} - \rho F + \nabla P\right] d\tau = 0$$